Activity

3-6 Theorems about Roots of Polynomial Equations

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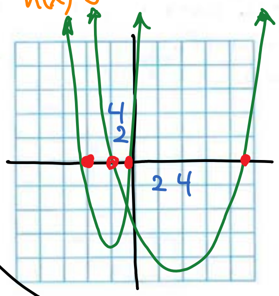
CRITIQUE & EXPLAIN

Look at the polynomial functions shown.

$g(x) = x^2 - 7x - 18$

$h(x) = 5x^2 + 24x + 16$?



A. Avery has a conjecture that the zeros of a polynomial function have to be positive or negative factors of its constant term. Factor g completely. Are the zeros of g factors of -18?

$(x-9)(x+2)=0$

$x-9=0 \quad | \quad x+2=0$

$x=9 \quad | \quad x=-2$

ac: (1)(-18)

Factors of -18

B. **Look for Relationships** Now test Avery's conjecture by factoring $h(x)$. Does Avery's conjecture hold? If so, explain why. If not, make a new conjecture.

GC: Roots:

$-4, -0.8$

.... relationship to constant?

$h(x) = 5x^2 + 24x + 16$

$-\frac{4}{5}$ → 16 Hmmm...

HABITS OF MIND

Use Structure For the factored function $k(x) = (ax + b)(cx + d)$, what are the coefficients in the expanded expression? MP.7

$acx^2 + adx$
$+ bcx + bd$

F L

$\rightarrow acx^2 + \underbrace{(ad + bc)}_{O \,\&\, I}x + bd$ Constant

leading Coefficient

Rational Root Theorem

• GC... match possible root to one of the possible rational solutions (roots/zeros)

Notes

$P(x) = \boxed{a_n} x^n + a_{n-1} x^{n-1} + \dots + a_1 x + \boxed{a_0}$

leading coefficient

Constant

q: factor of the leading coefficient

p: factor of the constant term

y=0 : roots

If $P(x) = 0$ has any rational roots, then rational root ⇨ $\frac{p}{q}$

EXAMPLE 1 **Try It! Identify Possible Rational Solutions**

1. List all the possible rational solutions for each equation.

a. $4x^4 + 13x^3 - 124x^2 + 212x - 8 = 0$ ← Standard Form

q: leading coef.   p: constant

p: ±1, ±2, ±4, ±8

q: ±1, ±2, ±4

$\frac{p}{q}$: $\pm1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm2, \pm4, \pm8$ } 12 possible roots

b. $7x^4 + 13x^3 - 124x^2 + 212x - 45 = 0$

q: leading coef   p: constant

p: ±1, ±3, ±5, ±9, ±15, ±45

q: ±1, ±7

$\frac{p}{q}$: $\pm1, \pm\frac{1}{7}, \pm3, \pm\frac{3}{7}, \pm5, \pm\frac{5}{7}, \pm9, \pm\frac{9}{7}, \pm15, \pm\frac{15}{7}, \pm45, \pm\frac{45}{7}$

Fundamental Theorem of Algebra

If $P(x)$ has degree $n \geq 1$, then $P(x) = 0$ has exactly n solutions in the set of complex numbers.

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ex) If $P(x)$ has any factor of multiplicity m, count the factor m times...

→ $(x-3)^4 = 0$ has 4 solutions

→ $x = 3$ (bounces or turning points)

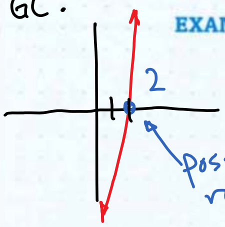
EXAMPLE 2 **Try It! Use the Rational Root Theorem**

2. A jewelry box measures $2x + 1$ in. long, $2x - 6$ in. wide, and x in. tall. The volume of the box is given by the function $v(x) = 4x^3 - 10x^2 - 6x$. What is the height of the box, in inches, if its volume is 28 in.^{3}?

HABITS OF MIND

Critique Arguments For the jewelry box, a student thought that the rational roots could be $\pm6, \pm3, \pm2, \pm1, \pm\frac{3}{2}, \pm\frac{1}{2}, \pm\frac{1}{4}$, using factors of −6 for the numerator and factors of 4 for the denominator of the possible rational roots. Is the student correct? Explain. MP.3

GC?



EXAMPLE 3 **Try It! Find All Complex Roots**

3. What are all complex roots of the equation $x^3 - 2x^2 + 5x - 10 = 0$?

q: ±1   degree   p: ±1, ±2, ±5, ±10

n:3 roots

+ | x

$\frac{p}{q}$: same list

root!

2	1	−2	5	−10
		2	0	10
	1	0	5	0

$x^2 + 5 = 0$

$x^2 = -5$

$\sqrt{x^2} = \pm\sqrt{-5}$

→ $x = \pm i\sqrt{5}$

→ $x = 2, i\sqrt{5}, -i\sqrt{5}$

also $\pm\sqrt{5}\, i$

Conjugate Root Theorem

... remember $a + \sqrt{b}$ &

Conjugate Root Theorem

- pairs
- $a+\sqrt{b}$ & $a-\sqrt{b}$
- $a+bi$ & $a-bi$

... remember $\frac{1}{3+\sqrt{2}} \rightarrow$ conjugate: $3-\sqrt{2}$

$a+\sqrt{b}$ &
$a-\sqrt{b}$

Notes

Assess

EXAMPLE 4 **Try It! Irrational Roots and the Coefficients of a Polynomial**

4. Suppose a quadratic polynomial function f has two complex zeros that are a conjugate pair, $a - bi$ and $a + bi$ (where a and b are real numbers). Are all the coefficients of f real? Explain.

Pairs...

HABITS OF MIND

Construct Arguments Could a polynomial equation with rational coefficients have two complex roots that are not conjugates as its only roots? Explain. **MP.3**

EXAMPLE 5 **Try It! Write Polynomial Functions Using Conjugates**

clue: x^2

5. a. What is a quadratic equation in standard form with rational coefficients that has a root of $5 + 4i$?

single root ... where's the other one? $\rightarrow$ conjugate of $5+4i$

$\boxed{(x-\text{root})}$

... $x = 5+4i$ & $5-4i$

$P(x) = (x-(5+4i))(x-(5-4i))$

$= (x-5-4i)(x-5+4i)$

$= x^2-5x+\cancel{4xi}-5x+25-\cancel{20i}-\cancel{4xi}+\cancel{20i}-16i^2$ $\quad i^2=-1$

$\boxed{P(x) = x^2-10x+41}$

b. What is a polynomial function Q of degree 4 with rational coefficients such that $Q(x) = 0$ has roots $2 - \sqrt{3}$ and $5i$?

conjugate: $-5i$

$2+\sqrt{3}$

$Q(x) = (x-5i)(x--5i)(x-(2+\sqrt{3}))(x-(2-\sqrt{3}))$

$= (x-5i)(x+5i)(x-2-\sqrt{3})(x-2+\sqrt{3})$

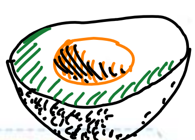
FOIL $(x^2-2x+\cancel{\sqrt{3}x}-2x+4-\cancel{2\sqrt{3}}-\cancel{\sqrt{3}x}+\cancel{2\sqrt{3}}-3)$

$= x^2-25i^2$ $\quad i^2=-1$

$= (x^2+25)(x^2-4x+1)$

$= x^4-4x^3+x^2$
$\quad +25x^2-100x+25$

$= \boxed{x^4-4x^3+26x^2-100x+25}$



HABITS OF MIND

Reason Is it possible to write a polynomial function of degree 3 that has rational coefficients and zeros $2 - \sqrt{3}$ and $5i$? Explain. **MP.2**

Need at least degree 4...

Assess

Do You UNDERSTAND?

ESSENTIAL QUESTION How are the roots of a polynomial equation related to the coefficients and degree of the polynomial?

1. **Error Analysis** Renaldo said that a polynomial equation with real coefficients has zeros $-1 + 2i$ and $3 + \sqrt{5}$ and has a degree of 4. Is Renaldo correct? Explain. MP.3

2. **Reason** A fifth degree polynomial $P(x)$ with rational coefficients has zeros $2i$ and $\sqrt{7}$. What other zeros does $P(x)$ have? Explain. MP.2

3. **Make Sense and Persevere** If one root of a polynomial equation is $4 + 2i$, is it certain that $4 - 2i$ is also a root of the equation? Explain. MP.1

Do You KNOW HOW?

List all the possible rational solutions for each equation according to the Rational Roots Theorem. Then find all of the rational roots.

4. $0 = x^3 + 4x^2 - 9x - 36$

5. $0 = x^4 - 2x^3 - 7x^2 + 8x + 12$

6. $0 = 4x^3 + 8x^2 - x - 2$

8 7. $0 = 9x^4 - 40x^2 + 16$

$p: \pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

$q: \pm 1, \pm 3, \pm 9$

$\frac{p}{q}: \pm 1, \pm\frac{1}{3}, \pm\frac{1}{9}, \pm 2, \pm\frac{2}{3}, \pm\frac{2}{9}, \pm 4, \pm\frac{4}{3}, \pm\frac{4}{9}, \pm 8, \pm\frac{8}{3}, \pm\frac{8}{9}, \pm 16, \pm\frac{16}{3}, \pm\frac{16}{9}$

GC? root

$$\begin{array}{r|rrrrr} 2 & 9 & 0 & -40 & 0 & 16 \\ & & 18 & 36 & -8 & -16 \\ \hline -2 & 9 & 18 & -4 & -8 & 0 \\ & & -18 & 0 & 8 & \\ \hline & 9 & 0 & -4 & 0 & \\ & x^2 & x & C & R & \end{array}$$

$9x^2 - 4 = 0$

$(3x-2)(3x+2) = 0$

$x = \frac{2}{3} \mid x = -\frac{2}{3}$

A polynomial equation with rational coefficients has the given roots. List two more roots of each equation.

8. $1 + \sqrt{11}$ and $-3 + \sqrt{17}$

$1 - \sqrt{11}$ $\quad -3 - \sqrt{17}$

$x = 4, 9i, -\sqrt{2}$

$-9i \quad \sqrt{2}$

$P(x) = (x-4)(x-9i)(x+9i)(x+\sqrt{2})(x-\sqrt{2})$

$= (x-4)(x^2 - 81i^2)(x^2 - 2)$

$= (x-4)\underbrace{(x^2+81)(x^2-2)}$

$x^4 - 2x^2 + 81x^2 - 162$

$= (x-4)(x^4 + 79x^2 - 162)$

$= x^5 + 79x^3 - 162x - 4x^4 - 316x^2 + 648$

$= \boxed{x^5 - 4x^4 + 79x^3 - 316x^2 - 162x + 648}$

9. $5 + 12i$ and $-9 - 7i$

$5 - 12i$ $\quad -9 + 7i$

10. $12 + 5i$ and $6 - \sqrt{13}$

$12 - 5i$ $\quad 6 + \sqrt{13}$

11. $5 - 15i$ and $17 + \sqrt{23}$

$5 + 15i$ $\quad 17 - \sqrt{23}$